Refined similarity hypotheses in shell models of homogeneous turbulence and turbulent convection

Emily S. C. Ching,[1,2] H. Guo,[1] and T. S. Lo[1]
[1]*Department of Physics, The Chinese University of Hong Kong, Shatin, Hong Kong, China*
[2]*Institute of Theoretical Physics, The Chinese University of Hong Kong, Shatin, Hong Kong, China*
(Received 5 April 2008; published 6 August 2008)

A major challenge in turbulence research is to understand from first principles the origin of the anomalous scaling of velocity fluctuations in high-Reynolds-number turbulent flows. One important idea was proposed by Kolmogorov [J. Fluid Mech. **13**, 82 (1962)], which attributes the anomaly to variations of the locally averaged energy dissipation rate. Kraichnan later pointed out [J. Fluid Mech. **62**, 305 (1973)] that the locally averaged energy dissipation rate is not an inertial-range quantity and a proper inertial-range quantity would be the local energy transfer rate. As a result, Kraichnan's idea attributes the anomaly to variations of the local energy transfer rate. These ideas, generally known as refined similarity hypotheses, can also be extended to study the anomalous scaling of fluctuations of an active scalar, such as the temperature in turbulent convection. We examine the validity of these refined similarity hypotheses and their extensions to an active scalar in shell models of homogeneous turbulence and turbulent convection. We find that Kraichnan's refined similarity hypothesis and its extension are valid.

DOI: 10.1103/PhysRevE.78.026303 PACS number(s): 47.27.eb

I. INTRODUCTION

Much effort in turbulence research has been devoted to the study of the possible universal statistics of velocity fluctuations in turbulent flows at high Reynolds number. The seminal work of Kolmogorov in 1941 (K41) [1] introduced the idea of universal homogeneous and isotropic statistics in high-Reynolds-number turbulent flows and in particular deduced that the statistical moments of the velocity differences, $\delta u_r(\vec{x},t) \equiv |\vec{u}(\vec{x}+\vec{r},t)-\vec{u}(\vec{x},t)|$, where $\vec{u}$ is the velocity field, have power-law scaling with the separating distance $r=|\vec{r}|$,

$$\langle \delta u_r(\vec{x},t)^n \rangle \sim (\langle \epsilon \rangle r)^{n/3}, \tag{1}$$

when r is within the inertial range. Here $\langle \cdots \rangle$ denotes an ensemble average. The inertial range refers to the range of length scales that are smaller than those of the energy input and larger than those affected directly by molecular dissipation, and the energy dissipation rate per unit mass ϵ is given by

$$\epsilon(\vec{x},t) = \frac{\nu}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\left[\frac{\partial u_i(\vec{x},t)}{\partial x_j}+\frac{\partial u_j(\vec{x},t)}{\partial x_i}\right]^2, \tag{2}$$

where ν is the kinematic viscosity, $\vec{x}=(x_1,x_2,x_3)$, and $\vec{u}=(u_1,u_2,u_3)$. The direct proportionality of the scaling exponents $n/3$ with the order n of the statistical moments is equivalent to an independence of r of the functional form of the probability density function (PDF) of δu_r. Experimental observations confirm the power-law scaling, but indicate that the scaling exponents depend on the order n in a nonlinear fashion. This deviation of the velocity scaling behavior from the K41 results is known as anomalous scaling and implies that turbulent velocity fluctuations have scale-dependent statistics and are thus intermittent.

A major challenge is to understand, from first principles, the origin of anomalous scaling. In his refined theory in 1962 [2], Kolmogorov replaced the global average energy dissipation rate $\langle \epsilon \rangle$ by the locally averaged energy dissipation rate ϵ_r, given by

$$\epsilon_r(\vec{x},t) = \frac{3}{4\pi r^3}\int_{|\vec{h}|\le r} \epsilon(\vec{x}+\vec{h},t)d\vec{h}. \tag{3}$$

The possible dependence of $\langle \epsilon_r^{n/3} \rangle$ on r allows for a correction to the $n/3$ scaling (see Sec. II). As a result, this idea of Kolmogorov, which we refer to as Kolmogorov's refined similarity hypothesis (RSH), attributes the origin of anomalous scaling to the variations of the local energy dissipation rate. Kraichnan [3] later pointed out that the local energy dissipation rate ϵ_r is not an inertial-range quantity and a proper inertial-range quantity would be the local energy transfer rate $\Pi_r(\vec{x},t)$ (see Sec. II for the precise definition), which measures the rate of energy transfer into scales smaller than r at a local point $\vec{x}$ in space. In other words, Kraichnan's idea, which we refer to as Kraichnan's RSH, attributes the origin of anomalous scaling to the variations of the local energy transfer rate. In a statistically steady state, for r in the inertial range, $\langle \Pi_r \rangle = \langle \epsilon \rangle = \langle \epsilon_r \rangle$, but $\Pi_r(\vec{x},t) \neq \epsilon_r(\vec{x},t)$. Hence the two RSHs of Kolmogorov and Kraichnan would give different results for the anomalous scaling exponents of velocity structure functions.

Anomalous scaling behavior has also been observed in the statistics of a scalar field advected by a turbulent velocity field. A passive scalar leaves the velocity statistics intact, while an active scalar couples with the velocity and influences its statistics. The nonlinear problem of anomalous scaling of active scalars, like that of velocity, remains unsolved. A common example of an active scalar is temperature in turbulent convection in which temperature variations drive the flow. Both Kolmogorov's and Kraichnan's RSH can be extended to turbulent convection to account for the anomalous scaling behavior of an active scalar.

In this paper, we examine the validity of Kolmogorov's and Kraichnan's RSH and their extensions to turbulent convection in shell models of turbulence. Shell models for homogeneous and isotropic turbulence have proved to be very successful in reproducing many of the statistical features of turbulent flows and in which high Reynolds number can be achieved with relative ease [4]. Shell models for homogeneous turbulent convection have also been studied. We use two shell models: the Sabra model for homogeneous and isotropic turbulence [5], which is an improved version of the so-called Gledzer-Ohkitani-Yamada (GOY) model [6,7], and the Brandenburg model for homogeneous turbulent convection [8].

The rest of this paper is organized as follow. We first summarize the mathematical formulations of the RSH of Kolmogorov and Kraichnan and the previous studies that examined their validity in Sec. II. In Sec. III, we discuss how these RSH are extended to turbulent convection in which temperature is an active scalar. We describe the Sabra shell model for homogeneous and isotropic turbulence and the Brandenburg shell model for homogeneous turbulent convection in Secs. IV and V, respectively. In Sec. VI, we examine the validity of the RSH of Kolmogorov and Kraichnan for homogeneous and isotropic turbulence in the Sabra model and the validity of their extensions to turbulent convection in the Brandenburg model. Finally, we conclude in Sec. VII.

II. KOLMOGOROV'S AND KRAICHNAN'S REFINED SIMILARITY HYPOTHESES

Kolmogorov's RSH can be stated as

$$\delta u_r(\vec{x},t) = \hat{\phi}[\epsilon_r(\vec{x},t)r]^{1/3}, \tag{4}$$

where $\hat{\phi}$ is a random variable independent of r and statistically independent of ϵ_r when the Reynolds number is much larger than 1. Equation (4) implies that

$$\langle(\delta u_r)^n\rangle \sim \langle\epsilon_r^{n/3}\rangle r^{n/3}. \tag{5}$$

For homogeneous flows, $\langle\epsilon_r(\vec{x},t)\rangle = \langle\epsilon(\vec{x},t)\rangle$ and is independent of r and $\vec{x}$. For $n\neq 3$, $\langle\epsilon_r^{n/3}\rangle$ generally depends on r and this r dependence allows for a correction to the K41 scaling. Thus Kolmogorov's RSH attributes the origin of anomalous scaling to the variations of the local energy dissipation rate.

Similarly, Kraichnan's RSH can be stated as

$$\delta u_r(\vec{x},t) = \hat{\psi}[|\Pi_r(\vec{x},t)|r]^{1/3}, \tag{6}$$

where $\hat{\psi}$ is a random variable independent of r and statistically independent of Π_r. The local energy transfer rate Π_r was defined by Kraichnan [3] using banded Fourier series:

$$\Pi_r(\vec{x},t) = -\sum_{m=n+1}^{\infty} u_i^m P_{ij}^m(\vec{\nabla})\left[u_s\frac{\partial u_j}{\partial x_s}\right], \tag{7}$$

where $\vec{u}^n(\vec{x},t)$, defined by $\vec{u}(\vec{x},t)=\Sigma_{n=0}^{\infty}\vec{u}^n(\vec{x},t)$, is the contribution from all wave numbers in the band $2^{n-1}k_0<k<2^n k_0$ or $0<k<k_0$ for $\vec{u}^0(\vec{x},t)$. Here k_0 is the characteristic wave number of the largest length scale of the motion and $P_{ij}^n(\vec{\nabla})$ is a band-limited solenoidal projection operator, defined in the Fourier representation by $P_{ij}^n(\vec{k})=\delta_{ij}-k_ik_j/k^2$ when k is in the nth band and $=0$ otherwise.

Equation (6) then implies

$$\langle(\delta u_r)^n\rangle \sim \langle|\Pi_r|^{n/3}\rangle r^{n/3}, \tag{8}$$

and Kraichnan's RSH thus attributes the origin of anomalous scaling to the variations of the local energy transfer rate. As the scaling behavior of $\langle|\Pi_r|^{n/3}\rangle$ would be generally different from that of $\langle\epsilon_r^{n/3}\rangle$, the two RSH, Eqs. (4) and (6), would give different results for the anomalous scaling exponents of the velocity structure functions.

Kolmogorov's RSH has been widely used in discussions of the anomalous scaling of the velocity structure functions. There have been quite a number of experimental and numerical studies that examine the validity of Eq. (4). Most of these studies checked whether δu_r and ϵ_r are correlated. Statistical correlation between δu_r and ϵ_r or its one-dimensional surrogate which represents ϵ by $\nu(\partial u_x/\partial x)^2$ or $(\nu/2)(\partial u_y/\partial x)^2$ was indeed reported [9–13]. It was, however, noted [14,15] that at least part of such a statistical correlation results from kinematical constraints independent of Navier-Stokes dynamics. Moreover, it would be more direct to check the implication of Eq. (4) that the conditional velocity structure functions at fixed values of the ϵ_r would be given by

$$\langle(\delta u_r)^n|\epsilon_r\rangle \sim r^{n/3}\epsilon_r^{n/3}. \tag{9}$$

The simple scaling behavior of $r^{n/3}$ in Eq. (9) implies that the mathematical form of the conditional PDFs of δu_r at fixed values of ϵ_r would be independent of r. The conditional PDFs of the difference of one velocity component at fixed values of the one-dimensional surrogate of ϵ_r [9] or its slight modification [16,17] were indeed reported to be close to Gaussian. Explicit calculations of the conditional velocity structure functions (using the longitudinal velocity difference instead of the whole velocity difference) were also carried out using direct numerical simulations of isotropic turbulence [18], but these simulations were limited to moderate Reynolds numbers, making it difficult to draw definitive conclusions. In particular, a clear demonstration of the scaling behavior of $r^{n/3}$ at fixed ϵ_r or the dependence of $\epsilon_r^{n/3}$ at fixed r is still lacking.

On the other hand, the validity of Kraichnan's RSH is much less examined. In a high-resolution direct numerical simulation of isotropic turbulence [19], it was shown that the scaling exponents of $\langle|r\Pi_r|^{p/3}\rangle$ are close to those of the p-order transverse velocity structure functions. In this work, the local energy transfer rate Π_r was defined [20] in the physical space as the contraction $-\vec{\nabla}\bar{\vec{u}}:\tau$, where $\bar{\vec{u}}(\vec{x},t)$ is the low-pass-filtered velocity with scales less than r removed and $\tau(\vec{x},t)$ is the turbulent stress tensor from scales less than r removed by the filtering. Since the transverse velocity difference is more intermittent than the longitudinal velocity difference [21], this result implies that the scaling exponents of $\langle|r\Pi_r|^{p/3}\rangle$ are close to those of the p-order whole velocity structure functions and are thus consistent with Eq. (8). However, a direct examination of Eq. (6) by studying the behavior of the conditional velocity structure functions at fixed values of the local energy transfer rate is yet to be performed.

III. REFINED SIMILARITY HYPOTHESES FOR TURBULENT CONVECTION

In turbulent convection, the statistics of δu_r as well as those of the temperature difference $\delta T_r \equiv T(\vec{x}+\vec{r},t)-T(\vec{x},t)$, where $T(\vec{x},t)$ is the temperature field, are of interest. Buoyancy is expected to drive the dynamics, thus affecting the statistics of δu_r and δT_r, for length scales r greater than the Bolgiano scale [22,23]. When buoyancy is dominant, it was suggested [24] that the statistics are governed by a cascade of temperature variance, which is proportional to entropy in the Boussinesq approximation [25]. Thus the role of the energy dissipation rate ϵ is now played by the temperature (variance) dissipation rate or the entropy dissipation rate χ, given by

$$\chi = \kappa \sum_{i=1}^{3} \left(\frac{\partial T}{\partial x_i} \right)^2, \tag{10}$$

where κ is the thermal diffusivity of the fluid. In the same spirit of deriving Eq. (4), δu_r and δT_r are expressed as functions of αg, χ_r, and r only, where α is the volume expansion coefficient of the fluid, g is the acceleration due to gravity, and χ_r is the locally averaged entropy dissipation rate, given by

$$\chi_r(\vec{x},t) = \frac{3}{4\pi r^3} \int_{|\vec{h}|\leq r} \chi(\vec{x}+\vec{h},t) d\vec{h}. \tag{11}$$

As usual, the functional dependence is obtained by dimensional analysis and the results are

$$\delta u_r = \hat{\Phi}_u (\alpha g)^{2/5} [\chi_r(\vec{x},t)]^{1/5} r^{3/5}, \tag{12}$$

$$\delta T_r = \hat{\Phi}_T (\alpha g)^{-1/5} [\chi_r(\vec{x},t)]^{2/5} r^{1/5}. \tag{13}$$

Here $\hat{\Phi}_u$ and $\hat{\Phi}_T$ are random variables independent of r and statistically independent of χ_r. Equations (12) and (13) are extensions [26] of Kolmogorov's RSH [Eq. (4)] to turbulent convection where buoyancy is dominant, and they attribute the origin of the anomalous scaling to the variations of the local entropy dissipation rate [27]. In particular, the conditional velocity and temperature structure functions at fixed values of χ_r would have simple scaling behavior in r,

$$\langle (\delta u_r)^n | \chi_r \rangle \sim r^{3n/5}, \tag{14}$$

$$\langle |\delta T_r|^n | \chi_r \rangle \sim r^{n/5}, \tag{15}$$

given by that of Bolgiano and Obukhov (BO) [22,23].

To get the corresponding expressions for the extensions of Kraichnan's RSH [Eq. (6)] to turbulent convection, one replaces χ_r by the locally entropy transfer rate Π_r^θ, which is defined as the rate of entropy transfer into scales smaller than r at a local point $\vec{x}$ in space. The results thus read

$$\delta u_r = \hat{\Psi}_u (\alpha g)^{2/5} |\Pi_r^\theta(\vec{x},t)|^{1/5} r^{3/5}, \tag{16}$$

$$\delta T_r = \hat{\Psi}_T (\alpha g)^{-1/5} |\Pi_r^\theta(\vec{x},t)|^{2/5} r^{1/5}. \tag{17}$$

Here $\hat{\Psi}_u$ and $\hat{\Psi}_T$ are random variables independent of r and statistically independent of Π_r^θ. Equations (16) and (17) therefore attribute the origin of anomalous scaling to variations of the local entropy transfer rate. Their implications are that the conditional velocity and temperature structure functions at fixed values of Π_r^θ would have simple BO scaling:

$$\langle (\delta u_r)^n | \Pi_r^\theta \rangle \sim r^{3n/5}, \tag{18}$$

$$\langle |\delta T_r|^n | \Pi_r^\theta \rangle \sim r^{n/5}. \tag{19}$$

Turbulent convection is often investigated experimentally in Rayleigh-Bénard convection cells heated from below and cooled on top (see, e.g., [28–30] for a review). Such confined convective flows are highly inhomogeneous in which thermal and viscous boundary layers are present near the top and bottom of the cell, and coherent structures are present. It was argued [26] that the presence of buoyant flow structures known as plumes could affect the scaling behavior, causing BO scaling to be invalid. Indeed there were indications from direct numerical simulations [31] and analyses of experimental data [32] that the scaling behavior of the central region of confined turbulent convection is not well described by BO scaling plus corrections. We note that in Ref. [32], the statistics of $\delta u_\tau \equiv |\vec{u}(\vec{x},t+\tau)-\vec{u}(\vec{x},t)|$ and $\delta T_\tau \equiv T(\vec{x},t+\tau)-T(\vec{x},t)$, the temporal counterparts of δu_r and δT_r, were studied because only measurements taken as a function of time at a fixed point in space were available. The validity of Eq. (13) was explored [27] by studying the conditional statistics of δT_τ at fixed values of χ_r, estimated by $\chi_\tau \propto (1/\tau)\int_t^{t+\tau} \kappa(\partial T/\partial t')^2 dt'$. It was found that for scales larger than the Bolgiano scale, $\langle |\delta T_\tau|^n | \chi_\tau \rangle / (\langle |T_\tau|^2 | \chi_\tau \rangle)^{n/2}$ become independent of τ. This result indicates that the scale or τ dependence of the statistics of δT_τ can be attributed to variations of χ_τ and that the extension of Kolmogorov's RSH holds in the buoyancy-driven regime in turbulent Rayleigh-Bénard convection.

On the other hand, the Brandenburg shell model for homogeneous turbulent convection is, by construction, free of boundaries and thus plumes, and the scaling behavior was reported to be BO plus intermittent corrections [33]. In direct numerical simulations of two-dimensional homogeneous turbulent convection, approximate BO scaling was also reported [34–36]. In an earlier work by one of us (Ching) [37], it was shown that for the Brandenburg model, the intermittent corrections to BO scaling can be solely attributed to shell-to-shell variations of the entropy transfer rate. These results thus verify the validity of the extension of Kraichnan's RSH. We shall discuss these results in greater depth and also examine the validity of the extension of Kolmogorov's RSH in Sec. V.

IV. SHELL MODEL FOR HOMOGENEOUS AND ISOTROPIC TURBULENCE

The basic idea of a shell model is to consider the velocity variable u_n, which can be associated with the velocity difference δu_r with $r=1/k_n$, in discrete "shells" in Fourier space. Here $k_n=k_0\lambda^n$, with $n=0,1,\ldots,N-1$, is the wave number of the nth shell and λ is customarily taken to be 2. In the Sabra model [5], u_n is complex and satisfies the following equation of motion:

$$\frac{du_n}{dt}=i(ak_nu_{n+2}u_{n+1}^*+bk_{n-1}u_{n+1}u_{n-1}^*-ck_{n-2}u_{n-1}u_{n-2}) - \nu k_n^2u_n+f\delta_{n,0}, \tag{20}$$

where $f\delta_{n,0}$ is the forcing acting only on the first shell, ν is the kinematic viscosity, and u_n^* is the complex conjugate of u_n. We use $a=1$, $b=-0.5$, and $c=-0.5$. With this choice of parameters, the model satisfies both energy and helicity conservation in the inviscid limit.

Multiplying Eq. (20) by u_n^*, the complex conjugate of u_n, and taking the average of the resulting equation and its complex conjugate, we obtain the energy budget for the nth shell:

$$\frac{1}{2}\frac{d|u_n|^2}{dt}=F_u(k_n)-F_u(k_{n+1})-\nu k_n^2|u_n|^2+\mathrm{Re}(fu_0^*)\delta_{n,0}, \tag{21}$$

where

$$F_u(k_n)\equiv k_n\,\mathrm{Im}\left(u_{n-1}^*u_n^*u_{n+1}+\frac{1}{4}u_nu_{n-1}^*u_{n-2}^*\right). \tag{22}$$

Here "Re" and "Im" represent the real and imaginary parts, respectively. The physical meaning of the different terms on the right-hand side of Eq. (21) is clear. The third term is the rate of energy dissipation in the nth shell due to viscosity, the last term is the power input due to external forcing, and F_n is the rate of energy transfer from the $(n-1)$th shell to the nth shell. In the stationary state and in the inertial range at which the external forcing is not acting and energy dissipation is negligible, Eq. (21) gives

$$0\approx\langle F_u(k_n)\rangle-\langle F_u(k_{n+1})\rangle, \tag{23}$$

which implies that $\langle F_u(k_n)\rangle$ is independent of k_n in the inertial range, which is the statement of energy cascade. The ensemble averages are evaluated as time averages when the system is in the stationary state. In the shell model, the local energy transfer rate Π_r can thus be naturally identified as $F_u(k_n)$ with $k_n=1/r$:

$$\Pi_r\to F_u(k_n). \tag{24}$$

As for the local energy dissipation rate, it has to be defined accordingly in the shell model. From Eq. (21), the total energy dissipation rate (at time t) in the shell model is

$$\epsilon(t)=\sum_{n=0}^{N-1}\nu k_n^2|u_n|^2. \tag{25}$$

We define the analog of the local energy dissipation rate ϵ_n in the shell model as

$$\epsilon_r\to\epsilon_n\equiv\sum_{n'=n}^{N-1}\nu k_{n'}^2|u_{n'}|^2, \tag{26}$$

again with $k_n=1/r$. With this definition, we have the nice result of $\epsilon_n(t)$ approaching $\epsilon(t)$ in the limit of $n\to 0$ or equivalently in the limit of $r\to L$. Summation of Eq. (21) from $n>1$ to $N-1$ gives

$$\frac{1}{2}\frac{d}{dt}\sum_{n'=n}^{N-1}|u_n'|^2=F_u(k_n)-\epsilon_n. \tag{27}$$

Thus, in the stationary state, we have

$$\langle F_u(k_n)\rangle=\langle\epsilon_n\rangle, \tag{28}$$

but $F_u(k_n)\neq\epsilon_n$, and in general, the two quantities can have different statistical features.

The velocity structure functions $Q_p(k_n)$ are statistical moments of $|u_n|$ and $Q_p(k_n)$ scales with k_n with scaling exponents γ_p:

$$Q_p(k_n)\equiv\langle|u_n|^p\rangle\sim k_n^{-\gamma_p}. \tag{29}$$

As reported in Ref. [5], the values of γ_p are close to those obtained in experiments and deviate from the K41 values of $p/3$. In this work, we are interested in examining the validity of the RSH to account for this anomalous scaling behavior of the velocity structure functions. With $r=1/k_n$, the shell-model analog of Kolmogorov's RSH [Eq. (4)] is written as

$$u_n=\phi\epsilon_n^{1/3}k_n^{-1/3}, \tag{30}$$

where ϕ is a random variable independent of n and statistically independent of ϵ_n. Equation (30) implies that the conditional velocity structure functions at fixed values of ϵ_n would have simple K41 scaling:

$$\tilde{Q}_p(k_n)\equiv\langle|u_n|^p|\epsilon_n\rangle\sim k_n^{-p/3}. \tag{31}$$

For simplicity of notation, we suppress the dependence on ϵ_n in the conditional velocity structure functions $\tilde{Q}_p$. The notation for the other conditional structure functions will be simplified in the same fashion. Similarly, the shell-model analog of Kraichnan's RSH reads

$$u_n=\psi|F_u(k_n)|^{1/3}k_n^{-1/3}, \tag{32}$$

where ψ is a random variable independent of n and statistically independent of $F_u(k_n)$. One implication of Eq. (32) is that $\langle|u_n|^{3p}\rangle$ and $\langle|F_u(k_n)|^p\rangle k_n^{-p}$ have the same scaling behavior. This implication has indeed been confirmed in the GOY [38] and Sabra [5] models. From the definition of F_u [see Eq. (22)], it is not surprising that $|F_u(k_n)|$ would have the same scaling behavior as $k_n|u_n|^3$, and thus the above implication could well be a direct consequence of the definition of $F_u(k_n)$ in the shell model. However, we emphasize that Eq. (32) is not merely a statement of the relation of the scaling exponents of $\langle|u_n|^{3p}\rangle$ and $\langle|F_u(k_n)|^p\rangle$. In particular, it has another important implication; namely, the conditional velocity structure functions at fixed values of $F_u(k_n)$ would have simple K41 scaling

$$Q_p^*(k_n) \equiv \langle |u_n|^p | F_u(k_n) \rangle \sim k_n^{-p/3}. \qquad (33)$$

Equation (33) is nontrivial if u_n fluctuates even when $F_u(k_n)$ is fixed at some given value, which would be the case when ψ is a random variable with a certain probability distribution rather than a number of fixed value.

V. SHELL MODEL FOR HOMOGENEOUS TURBULENT CONVECTION

Homogeneous turbulent convection has been proposed [39] as a convective flow in a box, with periodic boundary conditions, driven by a constant temperature gradient along the vertical direction. In the Boussinesq approximation, the equations of motion read [40]

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} = -\vec{\nabla} p + \nu \nabla^2 \vec{u} + \alpha g \theta \hat{z}, \qquad (34)$$

$$\frac{\partial \theta}{\partial t} + \vec{u} \cdot \vec{\nabla} \theta = \kappa \nabla^2 \theta + \beta u_z, \qquad (35)$$

with $\vec{\nabla} \cdot \vec{u} = 0$. Here, p is the pressure divided by the density, $\theta = T - (T_0 - \beta z)$ is the deviation of temperature from a linear gradient $-\beta$, T_0 is the mean temperature of the fluid, and $\hat{z}$ is a unit vector in the vertical direction.

A shell model for homogeneous turbulent convection driven by a temperature gradient was proposed by Brandenburg [8]. In Brandenburg's shell model, the velocity and temperature variables are real. We denote the velocity variable as v_n, to distinguish it from u_n in the Sabra model, and the temperature variable as θ_n. The equations of motions are

$$\frac{dv_n}{dt} + \nu k_n^2 v_n = \alpha g \theta_n + A k_n (v_{n-1}^2 - \lambda v_n v_{n+1}) + B k_n (v_n v_{n-1} - \lambda v_{n+1}^2), \qquad (36)$$

$$\frac{d\theta_n}{dt} + \kappa k_n^2 \theta_n = \beta v_n + \tilde{A}(k_n v_{n-1} \theta_{n-1} - k_{n+1} v_n \theta_{n+1}) + \tilde{B}(k_n v_n \theta_{n-1} - k_{n+1} v_{n+1} \theta_{n+1}), \qquad (37)$$

where A, B, $\tilde{A}$, and $\tilde{B}$ are positive parameters. Earlier work showed that the scaling behavior depends only on the ratio B/A [8]. It has been recently shown [37] that buoyancy drives the dynamics and affects the statistics of v_n and θ_n such that the scaling behavior is BO plus intermittent corrections when B/A is greater than some critical value of about 2. When buoyancy is driving the dynamics, energy is transferred from small to large scales on average [8]. As a result, a linear damping term $-f_0 v_n \delta_{n,0}$ [33,41] has to be added to Eq. (36) for the system to achieve stationarity.

One might attempt to construct a shell model for homogeneous turbulent convection by extending the GOY or Sabra model to include also a set of temperature variables and a coupling between the velocity and temperature variables. Such a model was proposed, but the scaling behavior of the velocity is still given by K41 plus corrections [42,43] and is thus unaffected by the presence of the temperature, in contrast to the Brandenburg model. In Ref. [44], it was clarified that buoyancy is only acting as a large-scale driving force and thus does not affect the statistics directly in such shell models extended from the GOY or Sabra model. In the present work, we are interested in studying the anomalous scaling of an active scalar, so we restrict our study to the Brandenburg model in which buoyancy acts directly on most scales and temperature behaves as an active scalar.

In the Bousinessq approximation, entropy is proportional to the volume integral of the temperature variance. Entropy in the nth shell is, therefore, defined as $\mathcal{S}_n \equiv \theta_n^2/2$. By studying the entropy budget obtained from Eq. (37) upon multiplication by θ_n,

$$\frac{d\mathcal{S}_n}{dt} = F_\theta(k_n) - F_\theta(k_{n+1}) - \kappa k_n^2 \theta_n^2 + \beta v_n \theta_n, \qquad (38)$$

we get the rate of entropy transfer or entropy flux from $(n-1)$th to nth shell as

$$F_\theta(k_n) \equiv k_n (\tilde{A} v_{n-1} + \tilde{B} v_n) \theta_{n-1} \theta_n. \qquad (39)$$

In the stationary state and in the intermediate range in which $\langle v_n \theta_n \rangle$ and the entropy dissipation $\kappa k_n^2 \langle \theta_n^2 \rangle$ are negligible, Eq. (38) gives

$$0 \approx \langle F_\theta(k_n) \rangle - \langle F_\theta(k_{n+1}) \rangle. \qquad (40)$$

Thus $\langle F_\theta(k_n) \rangle$ is independent of k_n in the intermediate range of scales, which is the statement of entropy cascade. We naturally identify the local entropy transfer rate Π_r^θ as $F_\theta(k_n)$ with $k_n = 1/r$:

$$\Pi_r^\theta \to F_\theta(k_n). \qquad (41)$$

In the Brandenburg shell model, the total entropy dissipation rate (at time t) is

$$\chi(t) = \sum_{n=0}^{N-1} \kappa k_n^2 \theta_n^2. \qquad (42)$$

Thus, as in Eq. (26), we define the analog of the local entropy dissipation rate in the shell model, χ_n, as

$$\chi_r \to \chi_n \equiv \sum_{n'=n}^{N-1} \kappa k_{n'}^2 \theta_{n'}^2, \qquad (43)$$

again with $k_n = 1/r$.

The velocity and temperature structure functions, defined by

$$S_p(k_n) \equiv \langle |v_n|^p \rangle \sim k_n^{-\zeta_p}, \qquad (44)$$

$$R_p(k_n) \equiv \langle |\theta_n|^p \rangle \sim k_n^{-\xi_p}, \qquad (45)$$

have been studied recently [37] and found to have anomalous scaling behavior. We would like to examine whether this anomalous scaling can be understood using the RSH extended to turbulent convection. The shell-model analogs of the RSH of Kolmogorov extended to turbulent convection, Eqs. (12) and (13), are

$$v_n = \Phi_v (\alpha g)^{2/5} \chi_n^{1/5} k_n^{-3/5}, \qquad (46)$$

$$\theta_n = \Phi_\theta(\alpha g)^{-1/5}\chi_n^{2/5}k_n^{-1/5}, \quad (47)$$

which imply that

$$\tilde{S}_p(k_n) \equiv \langle |v_n|^p|\chi_n\rangle \sim k_n^{-3p/5}, \quad (48)$$

$$\tilde{R}_p(k_n) \equiv \langle |\theta_n|^p|\chi_n\rangle \sim k_n^{-p/5}. \quad (49)$$

Similarly the shell model analogs of the RSH of Kraichnan extended to turbulent convection, Eqs. (16) and (17), are [37]

$$v_n = \Psi_v(\alpha g)^{2/5}|F_\theta(k_n)|^{1/5}k_n^{-3/5}, \quad (50)$$

$$\theta_n = \Psi_\theta(\alpha g)^{-1/5}|F_\theta(k_n)|^{2/5}k_n^{-1/5}, \quad (51)$$

and they imply

$$S_p^*(k_n) \equiv \langle |v_n|^p|F_\theta(k_n)\rangle \sim k_n^{-3p/5}, \quad (52)$$

$$R_p^*(k_n) \equiv \langle |\theta_n|^p|F_\theta(k_n)\rangle \sim k_n^{-p/5}. \quad (53)$$

Equations (52) and (53) have been confirmed in Ref. [37], thus supporting the validity of the extension of Kraichnan's RSH to turbulent convection [Eqs. (50) and (51)] in the Brandenburg model. The validity of the extension of Kolmogorov's RSH to turbulent convection [Eqs. (46) and (47)] in the Brandenburg model will be examined and discussed in Sec. VI.

VI. RESULTS AND DISCUSSIONS

A. Validity of the RSH of Kolmogorov and Kraichnan in the Sabra model

We numerically integrate Eq. (20) using the fourth-order Runge-Kutta method. Starting with $u_n = xk_n^{-1/3}$, where x is a random variable, we evolve the equations in time for a short period of time. The results so obtained, with the phases randomized, are used as the initial data for the actual runs. Following Ref. [5], we take the external forcing f as a time-correlated noise with exponential correlation and correlation time τ such that it follows the evolution equation

$$f(t+dt) = f(t)E + \sigma\sqrt{-2(1-E^2)\log_{10}(\rho_1)}\exp(i2\pi\rho_2), \quad (54)$$

where $E=\exp(-dt/\tau)$, σ is the standard deviation of f, and ρ_1 and ρ_2 are two uniform random numbers between 0 and 1. We use $\sigma=0.01$, $\tau=1$, and $f(t=0)=5(1+i)\times 10^{-3}$.

We calculate the velocity structure functions Q_p and the conditional velocity structure functions $\tilde{Q}_p$ and Q_p^*, respectively, at fixed values of ϵ_n and $F_u(k_n)$. The statistics are collected by averaging over approximately 1000 eddy turnover times of the largest scales. The scaling behavior of the model does not depend on the value of k_0 and is also independent of the value of ν when ν is small enough. We have used several different values of N. For the results reported below, we have used $k_0=2$, $\nu=2\times 10^{-8}$ and $N=24$, and $\nu=3.2\times 10^{-7}$ and $N=21$. Some of the results for the velocity structure functions Q_p are shown in Fig. 1. We observe that they exhibit a nice power-law dependence on k_n for $3<n<16$. The scaling exponents γ_p obtained are plotted in Fig. 2. These values, which are in good agreement with the values reported in the literature [5], deviate from $p/3$, demonstrating clearly that the velocity structure functions exhibit anomalous scaling.

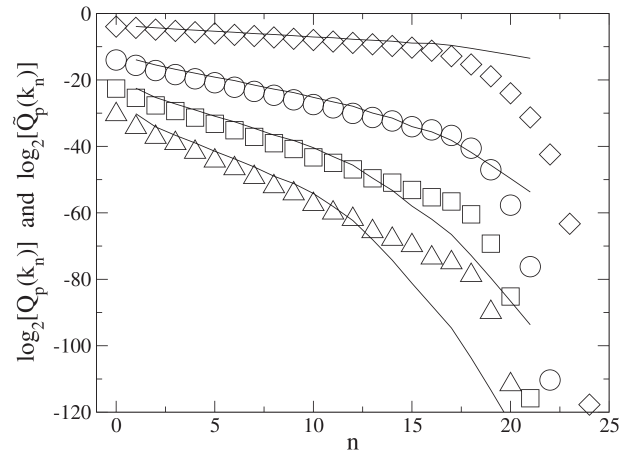

FIG. 1. The velocity structure functions $Q_p(k_n)$ as a function of the shell index n with $p=1$ (diamonds), $p=4$ (circles), $p=7$ (squares), and $p=10$ (triangles). The solid lines are the conditional velocity structure functions $\tilde{Q}_p(k_n)$ at given values of ϵ_n with $10^{-4} \leq \epsilon_n \leq 3\times 10^{-4}$ for the same values of p.

Our aim is to examine the validity of Kolmogorov and Kraichnan's RSH [Eqs. (30) and (32)] in accounting for the origin of the anomalous scaling behavior by examining the validity of Eqs. (31) and (33). To do so, we study the scaling exponents of $\tilde{Q}_p(k_n)$ and $Q_p^*(k_n)$, defined by

$$\tilde{Q}_p(k_n) \sim k_n^{-\tilde{\gamma}_p}, \quad (55)$$

$$Q_p^*(k_n) \sim k_n^{-\gamma_p^*}, \quad (56)$$

and check directly whether $\tilde{\gamma}_p$ and γ_p^* agree with $p/3$.

To calculate the conditional velocity structure functions $\tilde{Q}_p$ at fixed values of ϵ_n, we average only those $|u_n|^p$ when the value of ϵ_n is within a given narrow range of values. The

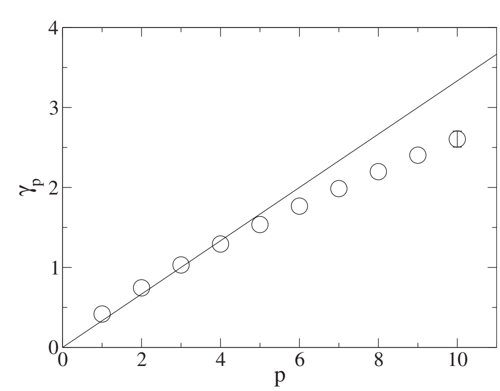

FIG. 2. The scaling exponents γ_p (circles) for $Q_p(k_n)$ as a function of p. The solid line is the K41 result of $p/3$. The error increases with p, and we show the largest error for $p=10$.

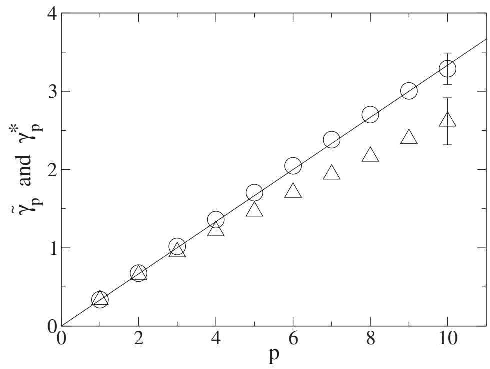

FIG. 3. The scaling exponents $\tilde{\gamma}_p$ (triangles) and γ^*_p (circles) of $\tilde{Q}_p(k_n)$ and $Q^*_p(k_n)$, respectively, as a function of p. The solid line is the K41 result of $p/3$, and the largest errors for $p=10$ are shown.

results for $\tilde{Q}_p$ are also shown in Fig. 1. It can be seen that $\tilde{Q}_p$ are different from Q_p. Moreover, the scaling range of $\tilde{Q}_p$ is shorter than that of Q_p. Thus $|u_n|$ is correlated with ϵ_n. Yet the scaling exponents $\tilde{\gamma}_p$ and γ_p are close to one another. In particular, as shown in Fig. 3, $\tilde{\gamma}_p$'s continue to deviate from $p/3$, demonstrating that Eq. (31) and thus Eq. (30) are invalid.

Similarly, to calculate the conditional velocity structure functions Q^*_p at fixed values of $F_u(k_n)$, we average only those $|u_n|^p$ when $F_u(k_n)$ assumes values in the same given narrow range. Some of the results are shown in Fig. 4. It can be seen that Q^*_p are again different from Q_p, thus showing that u_n and $F_u(k_n)$ are correlated. Moreover, the scaling range of Q^*_p is comparable to that of Q_p. In Fig. 3, we see that the scaling exponents γ^*_p are consistent with $p/3$. We also present the compensated plots of $Q^*_p(k_n)k_n^{p/3}$ in Fig. 5. It can be seen that the data points are consistent with being independent of n in the scaling range, thus demonstrating directly the scaling behavior of $Q^*_p \sim k_n^{-p/3}$. These results thus confirm Eq. (33).

As discussed in Sec. IV, it is important to check that u_n indeed fluctuates even when $F_u(k_n)$ is fixed at some given values. Thus we study the conditional probability density

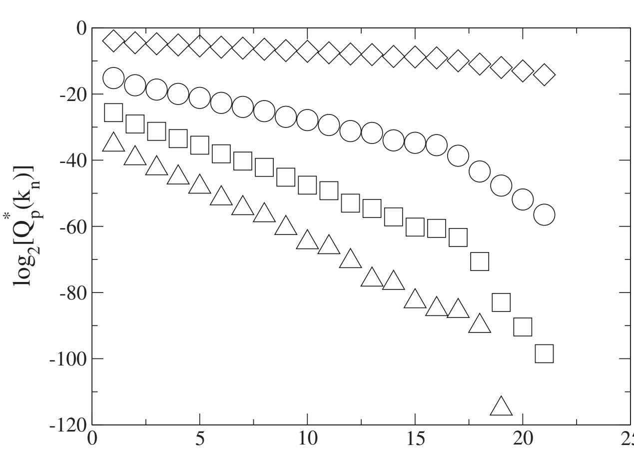

FIG. 4. The conditional velocity structure functions $Q^*_p(k_n)$ at fixed values of $F_u(k_n)$ with $10^{-4} \leqslant F_u(k_n) \leqslant 3\times10^{-4}$. Same symbols as in Fig. 1.

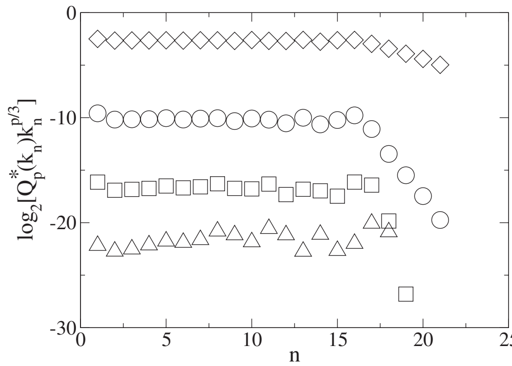

FIG. 5. Compensated plots of $Q^*_p(k_n)k_n^{p/3}$ versus n. Same symbols as in Fig. 4.

functions of u_n at fixed values of $F_u(k_n)$ and confirm that the distributions indeed have a finite extent (see Fig. 6). Moreover, the simple scaling behavior of Q^*_p implies that the conditional probability density functions $P(Y_n|F_u(k_n))$, with $Y_n \equiv |u_n|/\sqrt{\langle u_n^2|F_u(k_n)\rangle}$, are scale invariant or n independent. This is indeed the case as seen in Fig. 6. Our results thus show that Kraichnan's RSH [Eq. (32)] is valid in the Sabra shell model.

B. Validity of extensions of Kolmogorov and Kraichnan's RSH to turbulent convection in the Brandenburg model

We numerically integrate Eqs. (36) and (37) using the fourth-order Runge-Kutta method with an initial condition of $v_n=\theta_n=0$ except for a small perturbation of θ_n in an intermediate value of n. We use $k_0=1$, $A=0.01, B=1$, $\beta=1$, $\tilde{A}=\tilde{B}=1$, $\alpha g=1$, $\nu=5\times10^{-17}$, $\kappa=5\times10^{-15}$, $f_0=0.5$, and $N=32$. We calculate the velocity and temperature structure functions S_p and R_p when the system is in a stationary state. The results for S_p and R_p are shown in Figs. 7 and 8, respectively. It can be seen that they all have good scaling behavior.

In an earlier study [37], we have studied the scaling behavior of an active scalar using the Brandenburg model and found that ζ_p and ξ_p are given by the BO values plus corrections (see Fig. 9). This shows that S_p and R_p have anomalous

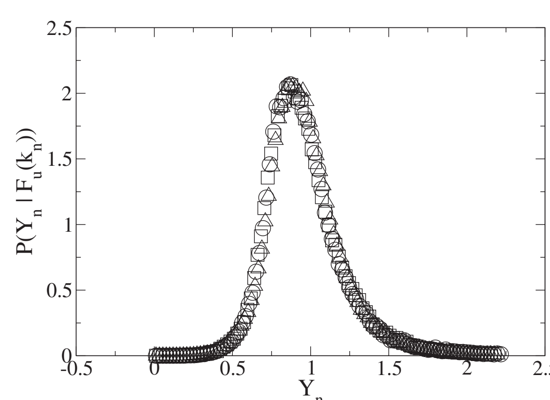

FIG. 6. The conditional probability density functions $P(Y_n|F_u(k_n))$ for $n=13$ (circles), $n=15$ (squares), and $n=17$ (triangles).

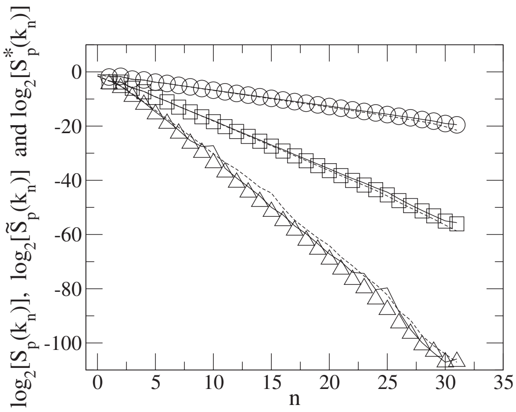

FIG. 7. S_p for $p=1$ (circles), $p=3$ (squares), and $p=6$ (triangles). The solid and dashed lines are, respectively, $\tilde{S}_p$ and S^*_p for the same values of p.

scaling behavior. In the same study, we have checked directly that the scaling exponents of S^*_p and R^*_p, denoted by ζ^*_p and ξ^*_p, have BO values of $3p/5$ and $p/5$, respectively, thus confirming the validity of Eqs. (52) and (53). To calculate S^*_p and R^*_p, we average $|v_n|^p$ and $|\theta_n|^p$ only when $F_\theta(k_n)=0.2\pm 10^{-4}$. We show S^*_p also in Fig. 7. It can be seen that S^*_p are close to S_p, but with a slightly steeper decrease with k_n. Results for R^*_p are shown in Fig. 10 in which the steeper decrease with k_n can be seen more clearly. The validity of Eqs. (52) and (53) indicates the validity of the extensions of Kraichnan's RSH to turbulent convection in the Brandenburg model.

With these results, we expect that the extensions of Kolmogorov's RSH to turbulent convection would be invalid in the Brandenburg model. In particular, we expect Eqs. (48) and (49) to be invalid. To show this, we calculate $\tilde{S}_p$ and $\tilde{R}_p$ by averaging $|v_n|^p$ and $|\theta_n|^p$ only when $\chi_n=0.2\pm 10^{-4}$. Results for $\tilde{S}_p$ and $\tilde{R}_p$ are shown also in Figs. 7 and 8, respectively. We find that both $\tilde{S}_p$ and $\tilde{R}_p$ are very close to S_p and R_p with $\tilde{R}_p$ having a shorter scaling range than R_p. The scaling exponents of $\tilde{S}_p$, denoted by $\tilde{\zeta}_p$, are close to ζ_p, which are close to $3p/5$ within errors. So we focus on the scaling exponents $\tilde{\xi}_p$ of $\tilde{R}_p$. In Fig. 11, we compare $\tilde{\xi}_p$ with ξ^*_p and the BO values of $p/5$ and see clearly that, as expected, Eq. (49) does not hold.

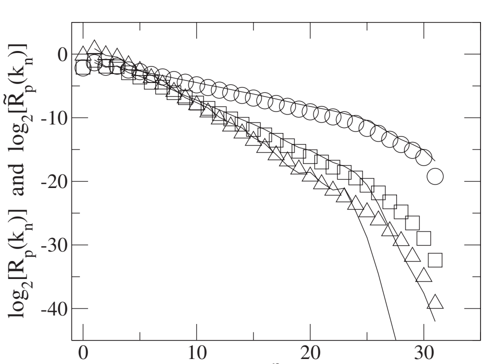

FIG. 8. R_p for $p=2$ (circles), $p=5$ (squares), and $p=8$ (triangles). The solid lines are $\tilde{R}_p$ for the same values of p.

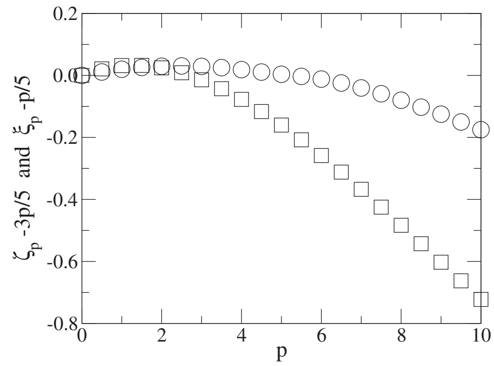

FIG. 9. The deviations of ζ_p and ξ_p from the BO values: $\zeta_p-3p/5$ (circles) and $\xi_p-p/5$ (squares).

VII. CONCLUSIONS

One long-standing goal in turbulence research is to understand, from first principles, the origin of the anomalous scaling of the fluctuating physical quantities. An important idea was proposed by Kolmogorov in his refined theory [2], which attributes the origin of the anomalous scaling behavior of the velocity fluctuations in homogeneous and isotropic turbulence to variations of the local energy dissipation rate. It was pointed out later by Kraichnan [3] that it is more appropriate to replace the local energy dissipation rate by the local energy transfer rate. We refer to these two ideas as Kolmogorov's and Kraichnan's RSH. Both of them can be extended to account for the anomalous scaling behavior of the velocity and temperature fluctuations in turbulent convection in which temperature acts as an active scalar. Specifically, in the extension of Kolmogorov's RSH to turbulent convection, the local entropy (or temperature variance) dissipation rate plays the role of the local energy dissipation rate. Similarly, in the extension of Kraichnan's RSH to turbulent convection, the local entropy transfer rate plays the part of the local energy transfer rate.

In this paper, we have examined the validity of Kolmogorov's and Kraichnan's RSH and their extensions to turbulent convection, respectively, in the Sabra shell model of homogeneous and isotropic turbulence [5] and in the Brandenburg shell model of homogeneous turbulent convection

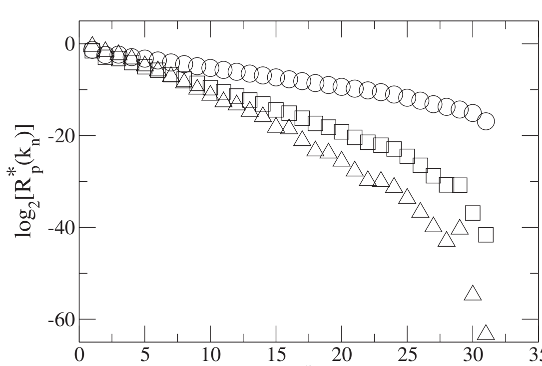

FIG. 10. The conditional temperature structure functions R^*_p at fixed values of $F_\theta(k_n)$ for $p=2$ (circles), $p=5$ (squares), and $p=8$ (triangles).

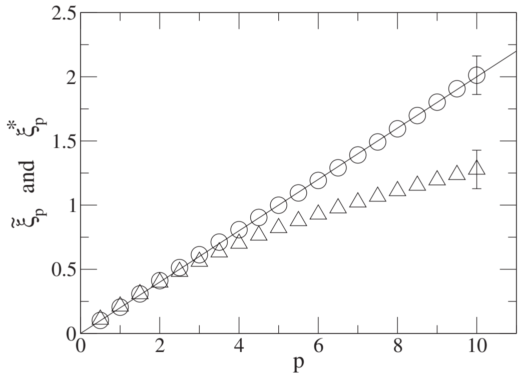

FIG. 11. Comparison of $\tilde{\xi}_p$ (triangles) and ξ^*_p (circles) with the BO values of $p/5$ (solid line). The largest errors for $p=10$ are shown.

[8]. The validity of Kolmogorov's RSH has been examined in previous studies. These studies mainly focused on the statistical correlation of the velocity difference and the local energy dissipation rate and the relation between the scaling exponents of the velocity structure functions and the moments of the local energy dissipation. The validity of Kraichnan's RSH is much less studied, and again the focus is on the relation between the scaling exponents of the velocity structure functions and the moments of the local energy transfer rate. An important consequence of Kolmogorov's or Kraichnan's RSH is that the conditional velocity structure functions at given values of the local energy dissipation rate or the local energy transfer rate would have simple K41 scaling behavior. Similarly, an important consequence of the extensions of Kolmogorov's or Kraichnan's RSH to turbulent convection is that the conditional velocity and temperature structure functions at given local entropy dissipation rate or the local entropy transfer rate would have simple BO scaling behavior. Our approach is to check directly these consequences. In the shell models, the local energy or entropy transfer rate is easily identified with the shell-to-shell energy or entropy transfer rate. On the other hand, the local energy or entropy dissipation rate has to be defined accordingly. As shown in Fig. 3, the scaling exponents γ^*_p of the conditional velocity structure functions at a given shell-to-shell energy transfer rate indeed have the K41 values of $p/3$, while the scaling exponents $\tilde{\gamma}_p$ of the conditional velocity structure functions at a given shell-model analog of the local energy dissipation rate continue to deviate from the K41 values. This result shows that Kraichnan's RSH, but not Kolmogorov's RSH, holds in the Sabra model. Similarly, as shown in Fig. 11, we have found that the scaling exponents ξ^*_p of the conditional temperature structure functions at a given shell-to-shell entropy transfer rate have the BO values of $p/5$, while the scaling exponents $\tilde{\xi}_p$ of the conditional temperature structure functions at a given shell-model analog of the local entropy dissipation rate continue to deviate from the BO values. Our result thus shows that the extension of Kraichnan's RSH, but not Kolmogorov's RSH, to turbulent convection holds in the Brandenburg model. In summary, our work shows that Kraichnan's RSH and its extension to turbulent convection hold in shell models of turbulence. It would be interesting to perform a similar study in direct numerical simulations and in experimental investigations.

ACKNOWLEDGMENTS

We thank Roberto Benzi and Luca Biferale for stimulating discussions. This work is supported in part by the Hong Kong Research Grants Council (Grant No. CA05/06.SC01).

[1] A. N. Kolmogorov, Dokl. Akad. Nauk SSSR **30**, 9 (1941); reproduced in Proc. R. Soc. London, Ser. A **434**, 9 (1991).
[2] A. N. Kolmogorov, J. Fluid Mech. **12**, 82 (1962).
[3] R. H. Kraichnan, J. Fluid Mech. **62**, 305 (1974).
[4] See, for example, L. Biferale, Annu. Rev. Fluid Mech. **35**, 441 (2003).
[5] V. S. L'vov, E. Podivilov, A. Pomyalov, I. Procaccia, and D. Vandembroucq, Phys. Rev. E **58**, 1811 (1998).
[6] E. B. Gledzer, Dokl. Akad. Nauk SSSR **200**, 1043 (1973).
[7] M. Yamada and K. Ohkitani, J. Phys. Soc. Jpn. **56**, 4210 (1987).
[8] A. Brandenburg, Phys. Rev. Lett. **69**, 605 (1992).
[9] G. Stolovitzky, P. Kailasnath, and K. R. Sreenivasan, Phys. Rev. Lett. **69**, 1178 (1992).
[10] A. A. Praskovsky, Phys. Fluids A **4**, 2589 (1992).
[11] S. T. Thoroddsen and C. W. van Atta, Phys. Fluids A **4**, 2592 (1992).
[12] S. Chen, G. D. Doolen, R. H. Kraichnan, and Z.-S. She, Phys. Fluids A **5**, 458 (1993).
[13] A. Praskovsky, E. Praskovskaya, and T. Horst, Phys. Fluids **9**, 2465 (1997).
[14] G. Stolovitzky and K. R. Sreenivasan, Rev. Mod. Phys. **66**, 229 (1994).
[15] S. Chen, G. D. Doolen, R. H. Kraichnan, and L.-P. Wang, Phys. Rev. Lett. **74**, 1755 (1995).
[16] Y. Gagne, M. Marchand, and B. Castaing, J. Phys. II **4**, 1 (1994).
[17] A. Naert, B. Castaing, B. Chabaud, B. Hébral, and J. Peinke, Physica D **113**, 73 (1998).
[18] L.-P. Wang, S. Chen, J. G. Brasseur, and J. C. Wyngaard, J. Fluid Mech. **309**, 113 (1996).
[19] Q. Chen, S. Chen, G. L. Eyink, and D. D. Holm, Phys. Rev. Lett. **90**, 214503 (2003).
[20] G. L. Eyink, J. Stat. Phys. **78**, 335 (1995).
[21] S. Chen, K. R. Sreenivasan, M. Nelkin, and N. Cao, Phys. Rev. Lett. **79**, 2253 (1997).
[22] R. Bolgiano, J. Geophys. Res. **64**, 2226 (1959).
[23] A. M. Obukhov, Dokl. Akad. Nauk SSSR **125**, 1246 (1959).
[24] V. S. L'vov, Phys. Rev. Lett. **67**, 687 (1991).
[25] See, for example, Landau and Lifshitz, *Fluid Mechanics* (Pergamon Press, Oxford, 1987).
[26] E. S. C. Ching, Phys. Rev. E **75**, 056302 (2007).

[27] E. S. C. Ching and K. L. Chau, Phys. Rev. E **63**, 047303 (2001).
[28] E. D. Siggia, Annu. Rev. Fluid Mech. **26**, 137 (1994).
[29] S. Grossmann and D. Lohse, J. Fluid Mech. **407**, 27 (2000).
[30] L. P. Kadanoff, Phys. Today **54** (8), 34 (2001).
[31] R. Verzicco and R. Camussi, J. Fluid Mech. **477**, 19 (2003); R. Camussi and R. Verzicco, Eur. J. Mech. B/Fluids **23**, 427 (2004).
[32] E. S. C. Ching, K. W. Chui, X.-D. Shang, P. Tong, and K.-Q. Xia, J. Turbul. **5**, 27 (2004).
[33] E. Suzuki and S. Toh, Phys. Rev. E **51**, 5628 (1995).
[34] S. Toh and E. Suzuki, Phys. Rev. Lett. **73**, 1501 (1994).
[35] D. Biskamp, K. Hallatschek, and E. Schwarz, Phys. Rev. E **63**, 045302(R) (2001).
[36] A. Celani, A. Mazzino, and M. Vergassola, Phys. Fluids **13**, 2133 (2001).
[37] E. S. C. Ching and W. C. Cheng, Phys. Rev. E **77**, 015303(R) (2008).
[38] L. Kadanoff, D. Lohse, J. Wang, and R. Benzi, Phys. Fluids **7**, 617 (1995).
[39] V. Borue and S. A. Orszag, J. Sci. Comput. **12**, 305 (1995).
[40] D. Lohse and F. Toschi, Phys. Rev. Lett. **90**, 034502 (2003).
[41] A. Celani, T. Matsumoto, A. Mazzino, and M. Vergassola, Phys. Rev. Lett. **88**, 054503 (2002).
[42] Jiang Mingshun and Liu Shida, Phys. Rev. E **56**, 441 (1997).
[43] L. Biferale (private communications).
[44] E. S. C. Ching, H. Guo, and W. C. Cheng, Physica D **237**, 2009 (2008).